Resit — Group Theory (WIGT-07)

11-04-2019, 9:00h–12.00h

University of Groningen

Instructions

1. Write your name and student number on every page you hand in.
2. All answers need to be accompanied with an explanation or a calculation.
3. Your grade for this exam is $(P+10)/10$, where P is the number of points for this exam.

Problem 1 (15 points)

a) Give the definition of the kernel of a group homomorphism.

Solution: Let $\varphi\colon G \to G'$ be a group homomorphism. Then the kernel of φ is the set of all elements of G mapped by φ to the unit element of G'. (5 points)

b) Write down the homomorphism theorem.

Solution: If $\psi\colon G \to G'$ is a homomorphism of groups, then $H := \ker(\psi)$ is a normal subgroup of G and we have

$$G/H \cong \psi(G) \leq G'.$$

(5 points)

c) Give the definition of the orthogonal group $O(n)$, where $n \geq 1$ is an integer.

Solution: $\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^T A = I\}$. (5 points)

Problem 2 (8 points)

Let $\tau = (1\,4\,3\,2)(5\,6\,7\,8)(6\,8\,1\,5)(4\,5\,7) \in S_8$.

a) Compute the order of τ. We compute the decomposition of τ into disjoint cycles and find $\tau = (1\,6\,5\,8\,4\,7\,3\,2)$ (3 points). Hence $\mathrm{ord}(\tau) = 8$ (1 point), because the order of cycle is the length of the cycle. (1 point) (5 points in total)

b) Compute the sign of τ.

Solution: Using a), the sign is $(-1)^{8-1} = -1$ (3 points)

Problem 3 (16 points)

Let H be the subset of A_4 consisting of the identity and all products of disjoint 2-cycles.

(a) Show that H forms a subgroup of A_4.

Solution: We use the subgroup criterion. Obviously the product of (1) with any element of H is in H. Let $i, j, k, \ell \in \{1, 2, 3, 4\}$ be distinct. Then we have

$$(i\,j)(k\,\ell)(i\,k)(j\,\ell) = (i\,\ell)(j\,k).$$

This shows that the product of two nontrivial elements of H is again in H. The inverse of a 2-cycle is itself, so this also holds for the product of disjoint 2-cycles, since these commute. In particular, they are in H. Because $(1) \in H$, this shows that $H \leq A_4$ by the subgroup criterion. (5 points)

(b) Show that $H \leq A_4$ is normal.

Solution: We need to show that $\tau\sigma\tau^{-1} \in H$ for any $\sigma \in H$ and arbitrary $\tau \in A_4$. This is trivial for $\sigma = (1)$, so let $\sigma := (i\,j)(k\,\ell) \in H$ with i, j, k, ℓ distinct.

We know that

$$\tau\sigma\tau^{-1} = \tau(i\,j)\tau^{-1}\tau(j\,\ell)\tau^{-1} = (\tau(i)\,\tau(j))(\tau(k)\,\tau(\ell)).$$

Now τ is a bijection, so $\tau(i), \tau(j), \tau(k), \tau(\ell)$ are distinct, and hence $\tau\sigma\tau^{-1} \in H$. (6 points)

(c) Show that A_4/H is an abelian group.

Solution: Since $H \leq A_4$ is normal, there is a group structure on A_4/H. We have $H = \{(1), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$, so H has order 4. Therefore A_4/H has order $|A_4|/4 = 3$. But every group of order 3 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and hence abelian. (5 points)

Problem 4 (15 points)

Let G be a group of order p^2q, where p and q are distinct prime numbers. Show that G is not simple. (Hint: You will need a case distinction.)

Solution: For a prime $\ell \mid p^2q$, let $n_p = n_p(G)$ be the number of Sylow-p groups in G. If we find $n_\ell = 1$ for some ℓ, then we know that the unique Sylow ℓ-group in G is normal and, since it has order ℓ, it is not G or $\{e\}$, so G is not simple. (2 points)

Suppose $p > q$. Then by Sylow's theorem $n_p \mid q$, so $n_p \in \{1, q\}$. (1 point) But $n_p \equiv 1 \pmod{p}$, (1 point) so $p \mid n_p - 1$, which is impossible for $n_p = q$ since $p > q$. Hence $n_p = 1$ and we are done. (3 points)

Now suppose $q < p$. Then $n_q \mid p^2$, so $n_q \in \{1, p, p^2\}$. (1 point) As above, we can't have $n_q = p$. (2 points) But if $n_q = p^2$, then there are p^2 subgroups of G of order q. Since q is prime, the intersection of two distinct such groups contains precisely the unit element of G, and moreover there are $q - 1$ elements of order q in such a group. (2 points) Therefore G contains $p^2(q - 1) = |G| - p^2$ elements of order q. (1 point) By Sylow's theorem, there is at least one Sylow-p-group in G. Since such a group has order p^2, it follows that there is exactly one of them, and we're done. (2 points)

Problem 5 (18 points)

The *center* of a group G is the set

$$Z(G) := \{x \in G \,:\, xy = yx \;\text{ for all } y \in G\},$$

i.e. the set of elements of G which commute with *all* elements of G. The center of G is a subgroup of G (you do not have to prove this).

(a) Show that if G is a group, then $Z(G) \leq G$ is normal.

Solution: To show that $Z(G)$ is normal, it suffices to show that for $x \in Z(G)$ and $y \in G$ we have $yxy^{-1} \in Z(G)$. But in fact this is equal to x, which proves normality. (3 points)

(b) Show that the set

$$H := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,:\, a, b \in \mathbb{R},\, a \neq 0 \right\}$$

forms a group under matrix multiplication.

Solution: The identity matrix is obviously in H. (1 point) Let $M = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, N = \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \in H$. Then

$$MN = \begin{pmatrix} ac & ad + bc^{-1} \\ 0 & (ac)^{-1} \end{pmatrix} \tag{1}$$

so $MN \in H$. (2 points) A simple computation shows $M^{-1} = \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \in H$. (2 points) By the subgroup criterion, H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$, hence a group under multiplication. (1 point) (6 points in total)

(c) Find the center of H.

Solution. Let $M = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in Z(H)$. According to (1), this means that for every $N = \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \in H$, the relation

$$ad + bc^{-1} = bc + a^{-1}d$$

holds. (2 points) But if $b \neq 0$, this implies

$$\frac{a - a^{-1}}{b} = \frac{c - c^{-1}}{d},$$

which can't be correct for all N, since the left hand side is independent of the choice of N and the right hand side is not. (2 points)

Similarly, if $a \neq a^{-1}$, then we get

$$\frac{b}{a - a^{-1}} = \frac{d}{c - c^{-1}},$$

which can't be satisfied for all N for the same reason. (2 points)

Therefore we must have $b = 0$ and $a = a^{-1}$, so $\pm M$ is the identity matrix. (2 points) And indeed both of these matrices are in the center of H, since they commute with every 2×2-matrix. (1 point)

(9 points in total)

Problem 6 (18 points)

a) Compute the rank and elementary divisors of $\mathbb{Z}^2 \times \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^\times$.

Solution: The rank of $\mathbb{Z}^2 \times \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^\times$ is 2, because $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^\times$ is finite.

Note that $|(\mathbb{Z}/4\mathbb{Z})^\times| = 2$, because in $\mathbb{Z}/4\mathbb{Z}$ we have

$$\bar{1}^2 = \bar{1},\ \bar{2}\bar{a} \in \{\bar{0}, \bar{2}\},\ \bar{3}^2 = \bar{1}$$

where the second congruence holds because an even integer is never congruent to 1 modulo 4. Hence $(\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z}$, as this holds for every group of order 2. We find that $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so the elementary divisors are 2 and 4. (7 points in total)

b) Let $H \leq \mathbb{Z}^3$ be generated by $(1, 0, -3), (4, 5, 1)$ and $(2, -1, 0)$. Show that $\mathbb{Z}^3/H \cong \mathbb{Z}/43\mathbb{Z}$.

Solution: The easiest way to see this is to note that

$$\det \begin{pmatrix} 1 & 0 & -3 \\ 4 & 5 & 1 \\ 2 & -1 & 0 \end{pmatrix} = 43.$$

By a result from the lectures, we have $|\mathbb{Z}^3/H| = 43$. But since 43 is prime, every group of this order is isomorphic to $\mathbb{Z}/43\mathbb{Z}$.

Alternatively, one can apply the algorithm from the lectures to find that the only elementary divisor of H is 43. (11 points in total).

End of test (90 points)